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Spectrum-Generating Symmetries for the Superpotentials $A\cot\theta$ and $B\tanh y$

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Abstract The hierarchy of supersymmetric partner Schrödinger equations for the superpotentials $A\cot\theta$ and $B\tanh y$ with A and B as half-integer and negative integer numbers are solved. The number of bound states for given trigonometric and hyperbolic potentials are infinite and finite, respectively. In addition to the spectrum-generating corresponding to the standard supersymmetry which is based on shifting potential parameter, there exist three other different methods for generating the spectrum. The first method is based on supersymmetrizing two given models via infinite and finite number of their bound states. This is realized by the ladder operators which shift only quantum numbers. The second and third methods are based on supersymmetrizing any of the models via all bound states corresponding to hierarchy of their partner potentials. They are respectively realized via simultaneous increasing or decreasing of quantum number and the potential parameter, and also, increasing one of them while decreasing the other. Any of the second and third methods leads to introducing two different classes of the algebraic solutions for both models.

Keywords Supersymmetry in quantum mechanics · Superpotentials · Shape invariance symmetries · Factorization methods · Special functions · Ordinary differential equations

1 Introduction and mathematical foundation

The generalization of the factorization method of Darboux was first provided by Schrödinger [1–3] in the context of quantum mechanics. Infeld and Hull in their review article have shown a large variety of second-order differential equations with boundary conditions set in six different types of factorization [4]. There have been some recent attempts on their mutual connections [5, 6]. On other hand, the idea of supersymmetry in the context of quantum mechanics was first studied by Nicolai [7] and Witten [8] and later by Cooper and Freedman [9]. At this time, Gendenshtein forwarded the concept of shape invariance in the framework of

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supersymmetric quantum mechanics [10, 11]. According to his idea, the supersymmetric partner potential corresponding to an original potential has the same spatial functionality so that the suitable parameters are just shifted. He has shown that for any shape-invariant potential, quantum states of consecutive spectral can be calculated by algebraic method. Afterwards, it was realized that the shape invariance is a sufficient but not a necessary condition for the solvability [12]. Much work has been done to investigate shape invariance in the framework of supersymmetry (for example, see [13–26]). Also, realization of the simultaneous shape invariance symmetries with respect to two different parameters n and m from the viewpoint of mathematics for the associated Jacobi, hypergeometric and Laguerre functions, as well as the associated Bessel polynomials, has been considered in [25, 27–29].

In this paper, the supersymmetric solutions of trigonometric and hyperbolic superpotentials $A\cot\theta$ and $B\tanh y$ are obtained for the half-integer and negative integer values of A and B, respectively. Their corresponding partner potentials with the explicit forms as $A^2\cot^2\theta \pm A\csc^2\theta$ and $B^2\tanh^2 y \pm B\,\mathrm{sech}^2 y$ have infinite and finite number of bound states, respectively. Supersymmetry essentially guarantees a spectrum generating algebra for the partner potentials. Therefore, in addition to the well-known standard method of supersymmetric partner Hamiltonians, we study the hidden supersymmetry properties in order to algebraically generate spectrum that some of them have not been followed in [25]. For this purpose, we use the raising and lowering relations for the associated Legendre functions $P_l^m(x)$.

As it is known, the associated Jacobi functions in the special case $\alpha=\beta=0$ are converted to the associated Legendre functions, which play the important role in many branches of theoretical and mathematical physics. In [25], it has been shown that the associated Jacobi functions $P_{n,m}^{(\alpha,\beta)}(x)$ ($\alpha,\beta>-1$, $-1\le x\le +1$, $n=0,1,2,\dots$ and $0\le m\le n$), satisfy the raising and lowering relations with respect to two parameters n and m simultaneously but separately. Now, one can easily extend the idea of [25] to the associated Legendre functions,

$$P_l^m(x)=\frac{A_l^m}{(1-x^2)^{\frac{m}{2}}}\left(\frac{d}{dx}\right)^{l-m}(1-x^2)^l$$
$$\text{with } A_l^m=\frac{(-1)^m}{2^{l+\frac{1}{2}}\Gamma(l+1)}\sqrt{\frac{(2l+1)\Gamma(l+m+1)}{\Gamma(l-m+1)}},\qquad l=0,1,2,\dots,-l\le m\le l, \tag{1}$$

by choosing zero values for both parameters α and β. Note that contrary to $0\le m\le n$, $-l\le m\le l$ is a symmetric interval. It is easy to see that the associated Legendre functions form an orthonormal set with the same m but with different l, that is,

$$\int_{-1}^{+1}P_l^m(x)P_{l'}^m(x)dx=\delta_{ll'}. \tag{2}$$

According to [25], the following laddering relations are simultaneously realized with respect to the parameters l and m, respectively:

$$A_+(l;x)P_{l-1}^m(x)=\sqrt{\frac{2l-1}{2l+1}(l^2-m^2)}\,P_l^m(x) \tag{3a}$$

$$A_-(l;x)P_l^m(x)=\sqrt{\frac{2l+1}{2l-1}(l^2-m^2)}\,P_{l-1}^m(x), \tag{3b}$$

and

$$A_+(m;x)P_l^{m-1}(x) = \sqrt{(l-m+1)(l+m)}P_l^m(x) \tag{4a}$$

$$A_-(m;x)P_l^m(x) = \sqrt{(l-m+1)(l+m)}P_l^{m-1}(x). \tag{4b}$$

The explicit forms of the laddering operators $A_\pm(l;x)$ and $A_\pm(m;x)$ are calculated as

$$A_\pm(l;x) = \pm(1-x^2)\frac{d}{dx} - lx, \tag{5}$$

$$A_\pm(m;x) = \pm\sqrt{1-x^2}\frac{d}{dx} + \frac{m-\frac{1}{2}\mp\frac{1}{2}}{\sqrt{1-x^2}}. \tag{6}$$

Simultaneous realization of laddering equations (3) and (4) with respect to l and m allows us to define the ladder operators shifting indices l and m simultaneously and agreeably, and shifting indices l and m simultaneously and inversely:

$$\begin{aligned} A_{\pm,\pm}(l,m;x) &:= \mp[A_\pm(l,x), A_\pm(m,x)] \\ &= \mp x\sqrt{1-x^2}\frac{d}{dx} - l\sqrt{1-x^2} - \frac{m-\frac{1}{2}\mp\frac{1}{2}}{\sqrt{1-x^2}}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} A_{\pm,\mp}(l,m;x) &:= \mp[A_\pm(l,x), A_\mp(m,x)] \\ &= \pm x\sqrt{1-x^2}\frac{d}{dx} + l\sqrt{1-x^2} - \frac{m-\frac{1}{2}\pm\frac{1}{2}}{\sqrt{1-x^2}}. \end{aligned} \tag{8}$$

The laddering relations (3) and (4) immediately give

$$A_{+,+}(l,m;x)P_{l-1}^{m-1}(x) = \sqrt{\frac{2l-1}{2l+1}(l+m-1)(l+m)}P_l^m(x) \tag{9a}$$

$$A_{-,-}(l,m;x)P_l^m(x) = \sqrt{\frac{2l+1}{2l-1}(l+m-1)(l+m)}P_{l-1}^{m-1}(x), \tag{9b}$$

and

$$A_{+,-}(l,m;x)\,P_{l-1}^m(x) = \sqrt{\frac{2l-1}{2l+1}(l-m)(l-m+1)}P_l^{m-1}(x) \tag{10a}$$

$$A_{-,+}(l,m;x)P_l^{m-1}(x) = \sqrt{\frac{2l+1}{2l-1}(l-m)(l-m+1)}P_{l-1}^m(x). \tag{10b}$$

The new formulation in terms of the associated Legendre functions allows us to realize the annihilation of the lowest bound states in the framework of supersymmetry idea for both models. Consequently, we can calculate algebraically all the bound states using the lowest states in four different methods.

2 Hierarchy of the Partner Potentials $A^2\cot^2\theta \pm A\csc^2\theta$

Using the function $f(x) = (1-x^2)^{-\frac{1}{4}}$ for performing a similarity transformation on the laddering relations (4), and also on (3), (9) and (10) along with applying the new variable θ,

$x=-\cos\theta$, $0\le\theta\le\pi$, we can obtain the four different types of laddering equations on the real wavefunctions $\Psi_l^m(\theta)=\sqrt{\sin\theta}\,P_l^m(-\cos\theta)$ as

$$A^{;m}_{,+}\Psi_l^{m-1}(\theta)=\sqrt{(l-m+1)(l+m)}\,\Psi_l^m(\theta) \tag{11a}$$

$$A^{;m}_{,-}\Psi_l^{m}(\theta)=\sqrt{(l-m+1)(l+m)}\,\Psi_l^{m-1}(\theta), \tag{11b}$$

$$A^{l;}_{+,}\Psi_{l-1}^{m}(\theta)=\sqrt{\frac{2l-1}{2l+1}(l^2-m^2)}\,\Psi_l^m(\theta) \tag{12a}$$

$$A^{l;}_{-,}\Psi_{l}^{m}(\theta)=\sqrt{\frac{2l+1}{2l-1}(l^2-m^2)}\,\Psi_{l-1}^m(\theta), \tag{12b}$$

$$A^{l,m}_{+,+}\Psi_{l-1}^{m-1}(\theta)=\sqrt{\frac{2l-1}{2l+1}(l+m-1)(l+m)}\,\Psi_l^m(\theta) \tag{13a}$$

$$A^{l,m}_{-,-}\Psi_{l}^{m}(\theta)=\sqrt{\frac{2l+1}{2l-1}(l+m-1)(l+m)}\,\Psi_{l-1}^{m-1}(\theta), \tag{13b}$$

$$A^{l,m}_{+,-}\Psi_{l-1}^{m}(\theta)=\sqrt{\frac{2l-1}{2l+1}(l-m)(l-m+1)}\,\Psi_l^{m-1}(\theta) \tag{14a}$$

$$A^{l,m}_{-,+}\Psi_{l}^{m-1}(\theta)=\sqrt{\frac{2l+1}{2l-1}(l-m)(l-m+1)}\,\Psi_{l-1}^{m}(\theta). \tag{14b}$$

The explicit forms of the raising and lowering operators can be calculated to give

$$\begin{aligned}
&A^{;m}_{,\pm}=\pm\frac{d}{d\theta}+W_m(\theta)\quad\text{with the superpotential } W_m(\theta)=\left(\frac{1}{2}-m\right)\cot\theta\\
&A^{l;}_{\pm,}=\pm\sin\theta\frac{d}{d\theta}+\left(l\mp\frac{1}{2}\right)\cos\theta\\
&A^{l,m}_{\pm,\pm}=\pm\cos\theta\frac{d}{d\theta}-\left(l\mp\frac{1}{2}\right)\sin\theta-\left(m-\frac{1}{2}\right)\csc\theta\\
&A^{l,m}_{\pm,\mp}=\mp\cos\theta\frac{d}{d\theta}+\left(l\mp\frac{1}{2}\right)\sin\theta-\left(m-\frac{1}{2}\right)\csc\theta.
\end{aligned} \tag{15}$$

The relations (11) describe the standard supersymmetry for the hierarchy of partner potentials (see [24])

$$V_{m,\pm}(\theta)=W_m^2(\theta)\pm\frac{dW_m(\theta)}{d\theta}=\left(m-\frac{1}{2}\right)^2\cot^2\theta\pm\left(m-\frac{1}{2}\right)\csc^2\theta, \tag{16}$$

via the following factorized Schrödinger equations ($\hbar=2M=1$) with respect to m

$$\begin{aligned}
&A^{;m}_{,+}A^{;m}_{,-}\Psi_l^m(\theta)=\left(-\frac{d^2}{d\theta^2}+V_{m,+}(\theta)\right)\Psi_l^m(\theta)=(l-m+1)(l+m)\Psi_l^m(\theta)\\
&A^{;m}_{,-}A^{;m}_{,+}\Psi_l^{m-1}(\theta)=\left(-\frac{d^2}{d\theta^2}+V_{m,-}(\theta)\right)\Psi_l^{m-1}(\theta)=(l-m+1)(l+m)\Psi_l^{m-1}(\theta).
\end{aligned} \tag{17}$$

The partner potentials are transformed into each other by the symmetry transformation $m \to -m$: $V_{-m,+}(\theta) = V_{m+1,-}(\theta)$. It is easy to conclude that the wavefunctions with the same m constitute an orthonormal set: $\int_0^\pi \Psi_l^{m*}(\theta)\Psi_{l'}^m(\theta)d\theta = \delta_{l\,l'}$. Also, the shape invariance condition (integrability condition) yields

$$V_{m,+}(\theta) - V_{m+1,-}(\theta) = 2m. \tag{18}$$

Thus, one can easily obtain the energy eigenvalues and wavefunctions. Whilst m is the parameter for partner potentials, different spectra are obtained by changing the label l. This means that l is a quantum number. Therefore, each of the partner potentials with a given m, have infinite number of bound states $\Psi_l^m(\theta)$ since, $l \geq m$. The relations (12), (13) and (14) show that the realization of solvability and supersymmetry can be followed without the need to rely on the supersymmetry of partner Hamiltonians described in (11).

Four different types of supersymmetry lead to the following methods of the spectrum-generating.

• *Spectrum-generating based on the standard supersymmetry of the partner Hamiltonians, shifting only potential parameter m.* Using (11a) and (11b), all wavefunctions $\Psi_l^m(\theta)$ are algebraically calculated as

$$\begin{aligned}
&\Psi_l^{\pm l}(\theta) = (-1)^{\frac{l\pm l}{2}} \frac{\sqrt{\Gamma(2l+2)}}{2^{l+\frac{1}{2}}\Gamma(l+1)} \sin^{l+\frac{1}{2}}\theta, \quad l \geq 0\\
&\Psi_l^m(\theta) = \sqrt{\frac{\Gamma(l\mp m+1)}{\Gamma(l\pm m+1)}}\, \frac{A_{,\pm}^{,m+\frac{1}{2}\mp\frac{1}{2}} A_{,\pm}^{,m+\frac{1}{2}\mp\frac{3}{2}} \cdots A_{,\pm}^{,\mp l+\frac{1}{2}\pm\frac{1}{2}} \Psi_l^{\mp l}(\theta)}{\sqrt{\Gamma(2l+1)}}\\
&\quad -l+1 \leq m \leq l,\ -l \leq m \leq l-1.
\end{aligned} \tag{19}$$

Contrary to the following three cases, the spectrum is generated by both of the raising and lowering operators. Note that in the algebraic method (19), the bound state $\Psi_l^m(\theta)$, corresponding to the potentials $V_{m,+}(\theta)$ and $V_{m+1,-}(\theta)$, is calculated by using $\Psi_l^l(\theta)$ and $\Psi_l^{-l}(\theta)$ which corresponds to the potentials $V_{l,+}(\theta)$ and $V_{l+1,-}(\theta)$ as well as to $V_{-l,+}(\theta)$ and $V_{-l+1,-}(\theta)$.

• *Spectrum-generating based on supersymmetry which shifts only the quantum number l.* Using (12a) and (12b), the following algebraic solutions for $\Psi_l^m(\theta)$'s are derived

$$\begin{aligned}
&\Psi_{|m|}^m(\theta) = (-1)^{\frac{m+|m|}{2}} \frac{\sqrt{\Gamma(2|m|+2)}}{2^{|m|+\frac{1}{2}}\Gamma(|m|+1)} \sin^{|m|+\frac{1}{2}}\theta, \quad -\infty < m < +\infty\\
&\Psi_l^m(\theta) = \sqrt{\frac{(2l+1)\Gamma(2|m|+1)}{\Gamma(l+m+1)\Gamma(l-m+1)}}\, \frac{A_{+,}^{l,} A_{+,}^{l-1,} \cdots A_{+,}^{|m|+1,} \Psi_{|m|}^m(\theta)}{\sqrt{2|m|+1}}, \quad l \geq |m|+1.
\end{aligned} \tag{20}$$

The infinite number of wavefunctions $\Psi_l^m(\theta)$ with $l \geq |m|$, are bound states corresponding to the partner potentials $V_{m,+}(\theta)$ and $V_{m+1,-}(\theta)$. The Schrödinger equations (17) show that for both of the potentials, $\Psi_{|m|}^m(\theta)$ is the ground state. Therefore, in this method the spectrum is generated by supersymmetrizing bound states of a given potential.

• *Spectrum-generating based on supersymmetry which shifts quantum number l and potential parameter m simultaneously and agreeably.* Simultaneous increase (decrease) in both the indices l and m in (13) allows us to define a positive new integer as $d := l - m + 1$, in the interest of eliminating l:

$$A_{+,+}^{d+m-1,m}\Psi_{d+m-2}^{m-1}(\theta) = \sqrt{\frac{2d+2m-3}{2d+2m-1}(d+2m-2)(d+2m-1)}\Psi_{d+m-1}^{m}(\theta) \quad (21a)$$

$$A_{-,-}^{d+m-1,m}\Psi_{d+m-1}^{m}(\theta) = \sqrt{\frac{2d+2m-1}{2d+2m-3}(d+2m-2)(d+2m-1)}\Psi_{d+m-2}^{m-1}(\theta). \quad (21b)$$

As d can be an odd or an even integer, i.e. $d = 2k-1$ or $d = 2k$ with $k = 1, 2, 3, \ldots$, the algebraic calculation of the spectrum is carried out in two different bunches in terms of states with the least values for m. One can easily conclude that $\Psi_{k-1}^{1-k}(\theta)$ and $\Psi_{k}^{1-k}(\theta)$ are respectively annihilated by $A_{-,-}^{k-1,1-k}$ and $A_{-,-}^{k,1-k}$. Therefore, the algebraic solutions of all bound states are settled in two different classes:

$$\begin{aligned}
&\Psi_{k-1}^{1-k}(\theta) = \frac{\sqrt{\Gamma(2k)}}{2^{k-\frac{1}{2}}\Gamma(k)}\sin^{k-\frac{1}{2}}\theta, \quad k = 1, 2, 3, \ldots \\
&\Psi_{2k+m-2}^{m}(\theta) = \frac{A_{+,+}^{2k+m-2,m}A_{+,+}^{2k+m-3,m-1}\cdots A_{+,+}^{k,2-k}\Psi_{k-1}^{1-k}(\theta)}{\sqrt{\frac{2k-1}{4k+2m-3}\Gamma(2k+2m-1)}}, \quad m > 1-k,
\end{aligned} \quad (22)$$

$$\begin{aligned}
&\Psi_{k}^{1-k}(\theta) = \frac{\sqrt{k\Gamma(2k+2)}}{2^{k}\Gamma(k+1)}\cos\theta\sin^{k-\frac{1}{2}}\theta, \quad k = 1, 2, 3, \ldots \\
&\Psi_{2k+m-1}^{m}(\theta) = \frac{A_{+,+}^{2k+m-1,m}A_{+,+}^{2k+m-2,m-1}\cdots A_{+,+}^{k+1,2-k}\Psi_{k}^{1-k}(\theta)}{\sqrt{\frac{2k+1}{4k+2m-1}\Gamma(2k+2m)}}, \quad m > 1-k.
\end{aligned} \quad (23)$$

Equations (17) show that the wavefunctions $\Psi_{k-1}^{1-k}(\theta)$ and $\Psi_{k}^{1-k}(\theta)$ are the ground and the first excited states of the both partner potentials $V_{1-k,+}(\theta)$ and $V_{2-k,-}(\theta)$. Therefore, supersymmetry corresponding to the simultaneous and agreeable shift of l and m leads to the calculation of bound states by using the ground and the first excited states with the least values for the potential parameter m.

• *Spectrum-generating based on supersymmetry which shifts quantum number l and potential parameter m simultaneously and inversely.* Let us define a positive new integer d' by $d' := l + m + 1$. Then, the relations (14) after transforming m to $m+1$ can be written in the form of lowering and raising equations with respect to the parameter m as below

$$A_{+,-}^{d'-m-1,m+1}\Psi_{d'-m-2}^{m+1}(\theta) = \sqrt{\frac{2d'-2m-3}{2d'-2m-1}(d'-2m-2)(d'-2m-1)}\Psi_{d'-m-1}^{m}(\theta) \quad (24a)$$

$$A_{-,+}^{d'-m-1,m+1}\Psi_{d'-m-1}^{m}(\theta) = \sqrt{\frac{2d'-2m-1}{2d'-2m-3}(d'-2m-2)(d'-2m-1)}\Psi_{d'-m-2}^{m+1}(\theta). \quad (24b)$$

Again, the (24) show that the annihilation of states depends on the odd or even values of d'. If we use the notations $d' = 2k-1$ and $d' = 2k$ with $k = 1, 2, 3, \ldots$, then it becomes easy to prove that the $\Psi_{k-1}^{k-1}(\theta)$ and $\Psi_{k}^{k-1}(\theta)$ are respectively annihilated by $A_{-,+}^{k-1,k}$ and $A_{-,+}^{k,k}$. By using relations (24), two different classes of the algebraic solutions corresponding to all bound states are obtained as follows

$$\begin{aligned}\Psi_{k-1}^{k-1}(\theta) &= \frac{(-1)^{k-1}\sqrt{\Gamma(2k)}}{2^{k-\frac{1}{2}}\Gamma(k)}\sin^{k-\frac{1}{2}}\theta, \quad k=1,2,3,\dots\\ \Psi_{2k-m-2}^{m}(\theta) &= \frac{A_{+,-}^{2k-m-2,m+1}A_{+,-}^{2k-m-3,m+2}\cdots A_{+,-}^{k,k-1}\Psi_{k-1}^{k-1}(\theta)}{\sqrt{\frac{2k-1}{4k-2m-3}\Gamma(2k-2m-1)}}, \quad m<k-1,\end{aligned} \tag{25}$$

$$\begin{aligned}\Psi_{k}^{k-1}(\theta) &= \frac{(-1)^{k-1}\sqrt{k\Gamma(2k+2)}}{2^{k}\Gamma(k+1)}\cos\theta\sin^{k-\frac{1}{2}}\theta, \quad k=1,2,3,\dots\\ \Psi_{2k-m-1}^{m}(\theta) &= \frac{A_{+,-}^{2k-m-1,m+1}A_{+,-}^{2k-m-2,m+2}\cdots A_{+,-}^{k+1,k-1}\Psi_{k}^{k-1}(\theta)}{\sqrt{\frac{2k+1}{4k-2m-1}\Gamma(2k-2m)}}, \quad m<k-1.\end{aligned} \tag{26}$$

The wavefunctions $\Psi_{k-1}^{k-1}(\theta)$ and $\Psi_{k}^{k-1}(\theta)$ are the ground and the first excited states of the both partner potentials $V_{k-1,+}(\theta)$ and $V_{k,-}(\theta)$. Consequently, supersymmetry corresponding to the simultaneous and inverse shift of l and m presents two different classes of algebraic solutions in terms of the ground and the first excited states with most values for the potential parameter m.

3 Hierarchy of the Partner Potentials $A^2\tanh^2 y \pm A\,\mathrm{sech}^2 y$

Using the change of variable $x=\tanh y$, $-\infty<y<+\infty$, we can get an orthonormality condition on the real wavefunctions $\Psi_l^m(y) := P_l^m(\tanh y)$ as $\int_{-\infty}^{+\infty}\Psi_l^m(y)\Psi_{l'}^m(y)\,\mathrm{sech}^2 y\,dy = \delta_{l\,l'}$. Then, the laddering relations (3), (4), (9) and (10) can be immediately written as

$$\mathcal{A}_{+,}^{l,}\Psi_{l-1}^{m}(y) = \sqrt{\frac{2l-1}{2l+1}(l^2-m^2)}\Psi_l^m(y) \tag{27a}$$

$$\mathcal{A}_{-,}^{l,}\Psi_{l}^{m}(y) = \sqrt{\frac{2l+1}{2l-1}(l^2-m^2)}\Psi_{l-1}^m(y), \tag{27b}$$

$$\mathcal{A}_{,+}^{;m}\Psi_{l}^{m-1}(y) = \sqrt{(l-m+1)(l+m)}\Psi_l^m(y) \tag{28a}$$

$$\mathcal{A}_{,-}^{;m}\Psi_{l}^{m}(y) = \sqrt{(l-m+1)(l+m)}\Psi_l^{m-1}(y), \tag{28b}$$

$$\mathcal{A}_{+,+}^{l,m}\Psi_{l-1}^{m-1}(y) = \sqrt{\frac{2l-1}{2l+1}(l+m-1)(l+m)}\Psi_l^m(y) \tag{29a}$$

$$\mathcal{A}_{-,-}^{l,m}\Psi_{l}^{m}(y) = \sqrt{\frac{2l+1}{2l-1}(l+m-1)(l+m)}\Psi_{l-1}^{m-1}(y), \tag{29b}$$

$$\mathcal{A}_{+,-}^{l,m}\Psi_{l-1}^{m}(y) = \sqrt{\frac{2l-1}{2l+1}(l-m)(l-m+1)}\Psi_l^{m-1}(y) \tag{30a}$$

$$\mathcal{A}_{-,+}^{l,m}\Psi_{l}^{m-1}(y) = \sqrt{\frac{2l+1}{2l-1}(l-m)(l-m+1)}\Psi_{l-1}^{m}(y), \tag{30b}$$

in which the explicit forms of the raising and lowering operators are

$$\begin{aligned}
\mathcal{A}^{l;}_{\pm,} &= \pm\frac{d}{dy} + W_l(y) \quad \text{with the superpotential } W_l(y) = -l\tanh y \\
\mathcal{A}^{;m}_{,\pm} &= \pm\cosh y\frac{d}{dy} + \left(m - \frac{1}{2} \mp \frac{1}{2}\right)\sinh y \\
\mathcal{A}^{l,m}_{\pm,\pm} &= \mp\sinh y\frac{d}{dy} - \left(m - \frac{1}{2} \mp \frac{1}{2}\right)\cosh y - l\,\mathrm{sech}\, y \\
\mathcal{A}^{l,m}_{\pm,\mp} &= \pm\sinh y\frac{d}{dy} - \left(m - \frac{1}{2} \pm \frac{1}{2}\right)\cosh y + l\,\mathrm{sech}\, y.
\end{aligned} \tag{31}$$

Now with the help of (27) one can obtain the Schrödinger equations ($\hbar = 2M = 1$) factorized with respect to l as

$$\begin{aligned}
\mathcal{A}^{l;}_{+,}\mathcal{A}^{l;}_{-,}\Psi^m_l(y) &= \left(-\frac{d^2}{dy^2} + V_{l,+}(y)\right)\Psi^m_l(y) = (l^2 - m^2)\Psi^m_l(y) \\
\mathcal{A}^{l;}_{-,}\mathcal{A}^{l;}_{+,}\Psi^m_{l-1}(y) &= \left(-\frac{d^2}{dy^2} + V_{l,-}(y)\right)\Psi^m_{l-1}(y) = (l^2 - m^2)\Psi^m_{l-1}(y),
\end{aligned} \tag{32}$$

so that, the partner potentials for the hyperbolic superpotential $W_l(y)$ have the following explicit forms

$$V_{l,\pm}(y) = W_l^2(y) \pm \frac{dW_l(y)}{dy} = l^2\tanh^2 y \mp l\,\mathrm{sech}^2\, y. \tag{33}$$

Note that $W_l(y)$ can be considered as a special case of the Rosen Morse II superpotential. The shape invariance equations (32) can be described as the following condition on the hierarchy of partner potentials

$$V_{l,+}(y) - V_{l+1,-}(y) = -(2l+1). \tag{34}$$

The Schrödinger equations (32) show that l is the parameter of partner potentials and m is a finite quantum number, since $-l \le m \le l$. In fact, each of partner potentials with a given l, has finite number of bound states $\Psi^m_l(y)$. Therefore, the shape invariance condition (34) determines the number of independent wavefunctions for given partner potentials $V_{l,+}(y)$ and $V_{l,-}(y)$. The realization of solvability and supersymmetry can also be followed in four different methods. Supersymmetry corresponding to the partner Hamiltonians, the so-called the standard supersymmetry, is realized by (27).

• *Spectrum-generating based on the standard supersymmetry of the partner Hamiltonians, shifting only potential parameter l.* By solving the first order differential equation $\mathcal{A}^{|m|;}_{-,}\Psi^m_{|m|}(y) = 0$, and also, with the successive application of laddering equation (27a), all bound states $\Psi^m_l(y)$ with $l \ge |m|$ can be obtained as

$$\begin{aligned}
\Psi^m_{|m|}(\theta) &= (-1)^{\frac{m+|m|}{2}}\frac{\sqrt{\Gamma(2|m|+2)}}{2^{|m|+\frac{1}{2}}\Gamma(|m|+1)}\,\mathrm{sech}^{|m|}\, y, \quad -\infty < m < +\infty \\
\Psi^m_l(y) &= \sqrt{\frac{(2l+1)\Gamma(2|m|+1)}{\Gamma(l+m+1)\Gamma(l-m+1)}}\,\frac{\mathcal{A}^{l;}_{+,}\mathcal{A}^{l-1;}_{+,}\cdots\mathcal{A}^{|m|+1;}_{+,}\Psi^m_{|m|}(y)}{\sqrt{2|m|+1}}, \quad l \ge |m|+1.
\end{aligned} \tag{35}$$

The Schrödinger equations (32) show that $\Psi^m_{|m|}(y)$ and $\Psi^m_l(y)$ are the bound states corresponding to the potentials $V_{|m|,+}(y)$ and $V_{|m|+1,-}(y)$ as well as to $V_{l,+}(y)$ and $V_{l+1,-}(y)$, respectively. Therefore, in this method the spectrum is generated by supersymmetrizing the hierarchy of the partner potentials.

• *Spectrum-generating based on supersymmetry which shifts only the quantum number m.* By making use of (28a) and (28b), one can algebraically calculate all quantum states as follows

$$\begin{aligned}
&\Psi^{\pm l}_l(\theta) = (-1)^{\frac{l\pm l}{2}} \frac{\sqrt{\Gamma(2l+2)}}{2^{l+\frac{1}{2}}\Gamma(l+1)} \operatorname{sech}^l y, \quad l \geq 0 \\
&\Psi^m_l(y) = \sqrt{\frac{\Gamma(l\mp m+1)}{\Gamma(l\pm m+1)}} \frac{\mathcal{A}^{,m+\frac{1}{2}\mp\frac{1}{2}}_{,\pm} \mathcal{A}^{,m+\frac{1}{2}\mp\frac{3}{2}}_{,\pm} \cdots \mathcal{A}^{,\mp l+\frac{1}{2}\pm\frac{1}{2}}_{,\pm} \Psi^{\mp l}_l(y)}{\sqrt{\Gamma(2l+1)}}, \\
&\quad -l+1 \leq m \leq l, \; -l \leq m \leq l-1.
\end{aligned} \tag{36}$$

Note that in the algebraic method (36), the bound states $\Psi^m_l(y)$ are generated via supersymmetrizing the finite quantum number m for given partner potentials.

• *Spectrum-generating based on supersymmetries which shift l and m simultaneously and agreeably as well as simultaneously and inversely.* The laddering equations corresponding to the shifting of parameters l and m, simultaneously and agreeably as well as simultaneously and inversely, are again realized by all the bound states situated on the lines $l = m + d - 1$ and $l = -m + d' - 1$, respectively. The simultaneous shift of both the indices l and m in (29) and (30) leads to equations similar to (21)–(26) for the hyperbolic partner potentials, with the ground and the first excited states different from these in below:

$$\begin{aligned}
\Psi^{1-k}_{k-1}(y) &= \frac{\sqrt{\Gamma(2k)}}{2^{k-\frac{1}{2}}\Gamma(k)} \operatorname{sech}^{k-1} y \\
\Psi^{1-k}_{k}(y) &= \frac{-\sqrt{k\Gamma(2k+2)}}{2^k\Gamma(k+1)} \sinh y \operatorname{sech}^k y \\
\Psi^{k-1}_{k-1}(y) &= \frac{(-1)^{k-1}\sqrt{\Gamma(2k)}}{2^{k-\frac{1}{2}}\Gamma(k)} \operatorname{sech}^{k-1} y \\
\Psi^{k-1}_{k}(y) &= \frac{(-1)^k\sqrt{k\Gamma(2k+2)}}{2^k\Gamma(k+1)} \sinh y \operatorname{sech}^k y.
\end{aligned} \tag{37}$$

Therefore, in this paper, we presented four different methods of supersymmetrizing both models in order to generate the spectrum of bound states.

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